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Chaos synchronization of nonlinear gyros in presence of stochastic excitation via sliding mode control

Hassan Salarieh, Aria Alasty*

Department of Mechanical Engineering, Center of Excellence in Design, Robotics, and Automation (CEDRA), Sharif University of Technology, Azadi Avenue, Tehran 11365-9567, Iran

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Abstract

In this paper the problem of synchronization between two chaotic gyros with stochastic base excitation, based on the mater–slave scheme, is studied. The stochastic excitation is modeled by applying a Gaussian white noise to the deterministic model of gyro. The white noise is derived from the Wiener process, so the resulted system is a stochastic differential equation and is modeled by an Ito differential form. Using a modified sliding mode control a Markov synchronizing control law is designed and the convergence of error states to the sliding surface in the mean square norm is analytically proved. Simulation results show the high performance of the proposed controller in stochastic chaos synchronization of gyro systems.

1. Introduction

In the last few years, synchronization in chaotic dynamical systems has received a great deal of interest among scientists from various fields [1,2]. The results of chaos synchronization are utilized in biology, chemistry, secret communication and cryptography, nonlinear oscillation synchronization and some other nonlinear fields. The first idea of synchronizing two identical chaotic systems with different initial conditions was introduced by Pecora and Carroll [3], and the method was realized in electronic circuits. The methods for synchronization of the chaotic systems have been widely studied in recent years, and many different methods have been applied theoretically and experimentally to synchronize chaotic systems [4], such as adaptive control [5,6], backstepping [7] and sliding mode control [8].

One of the most attractive dynamic systems is the gyroscope. Gyroscopes have great utility in many scientific and engineering areas, such as optics, aeronautics and navigation. A simple satellite is indeed a symmetric gyro and is used for communication. It has been proved that in special situations a gyro may show chaotic dynamics. The chaotic behavior of gyros was introduced originally in Ref. [9]. In Refs. [10,11], the nonlinear dynamics of a symmetric heavy gyroscope mounted on a vibrating platform was studied. In these works a linear damping coefficient was assumed for the gyro system. In Refs. [12,13], it was shown that under

*Corresponding author. Tel.: +98 21 6616 5586; fax: +98 21 6600 0021.
E-mail addresses: salarieh@mech.sharif.edu (H. Salarieh), aalasti@sharif.edu (A. Alasty).

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base harmonic excitation and a nonlinear damping force, the gyro system shows chaotic behavior. Recently, synchronization of chaotic gyros has been widely investigated by many researchers. Synchronization of two gyros is usually used in areas of secure communications [14], attitude control of long-duration spacecraft [15], and signal processing in optical gyros. Synchronization of two similar chaotic gyro models of Ref. [12] via active control was studied in Ref. [16]. In Ref. [17] the synchronization problem based on the adaptive sliding mode control method between the two chaotic gyro models of Ref. [12] was investigated. In that work the authors assume that the slave system is perturbed by a bounded uncertain signal, and the system parameters are not known.

All of the mentioned works have modeled the parameters and excitations of chaotic gyro systems in the deterministic form, but in the real world, due to random uncertainties such as stochastic forces or excitations and noisy measurements caused by environmental uncertainties, a stochastic chaotic behavior is produced instead of a deterministic one. In this case the deterministic differential equation of the system must be substituted by a stochastic differential equation. There are a few works in the field of stochastic chaos and its control or synchronization [18–20]. In this paper the problem of chaos synchronization in chaotic gyro systems is investigated. The dynamics under investigation are for a symmetrical gyro with nonlinear damping, which is subjected to a harmonic excitation [12,13] with a random perturbation produced by white Gaussian noise. It is also assumed that the parameters of the gyro model have some uncertainties, i.e. the exact values of the parameters are not known, while the nominal values are known. The white Gaussian noise is modeled through a standard Wiener process, so the deterministic chaotic response of the system is substituted by a stochastic chaotic one, which is modeled by an Ito differential form [21]. It must be noted that the bounded external disturbance, which was considered in Ref. [17] or [19], does not generate the standard stochastic differential equation [22], while in this paper a stochastic differential equation is produced in the standard form [21]. For modeling a stochastic chaotic system, similar to other standard stochastic differential equations, a white Gaussian noise generated by derivations of a Wiener process must be applied to a deterministic system [18].

In this paper after introducing the synchronization objective, the problem of stochastic chaos synchronization in a general class of uncertain systems, i.e. affine systems with uncertain parameters in their models, is investigated and the sliding mode technique is modified for this purpose. The sliding mode method has a well-known property of robustness against uncertainties [23]. In this paper, considering this property the conventional sliding mode method is modified and applied for synchronizing the chaotic dynamics of the gyro models presented in Ref. [12] with uncertain parameters and stochastic base excitations. Simulation results of applying the modified sliding mode control to the gyros dynamics show the effectiveness of the method.

2. Nonlinear gyro dynamics

The equation governing the motion of a symmetric gyro mounted on a vibrating base in terms of the nutation angle θ, i.e. the angle which the spin axis of the gyro makes with the vertical axis, is given by [12]

$$\ddot{\theta} + \alpha^2 \frac{(1-\cos\theta)^2}{\sin^3\theta} - \beta\sin\theta + c_1\dot{\theta} + c_2\dot{\theta}^3 = f\sin\omega t\sin\theta, \tag{1}$$

where $f\sin\omega t$ is a parametric excitation that models the base excitation, $c_1\dot{\theta}$ and $c_2\dot{\theta}^3$ are linear and nonlinear damping, respectively, and $\alpha^2((1-\cos\theta)^2/\sin^3\theta) - \beta\sin\theta$ is like a nonlinear resilience force; note that by linearizing it around $\theta = 0$ a form of $K\theta$, which is similar to a spring force, is obtained. According to Ref. [12], in a symmetric gyro mounted on a vibrating base, the precession and the spin angles have cyclic motions and hence their momentum integrals are constant and equal to each other. So the governing equations of motion depend only on the nutation angle. Using Routh's procedure and assuming a linear-plus-cubic form for dissipative force, Eq. (1) is obtained [12]. Under certain conditions the above model of the gyro shows a chaotic response [12,13]. Defining $x_1 = \theta$, $x_2 = \dot{\theta}$ and $g(x) = -\alpha^2((1-\cos x)^2/\sin^3 x)$, system (1) is rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= g(x_1) - c_1x_2 - c_2x_2^3 + \beta\sin(x_1) + f\sin\omega t\sin x_1. \end{aligned} \tag{2}$$

In Ref. [16] the synchronization of two chaotic gyros is investigated by applying two control actions to each of the above two equations. In that work, it is assumed that both the master and the slave systems are identical, and the initial conditions are not the same. In addition the system parameters are known without uncertainty. In this work, it is assumed that the system parameters have some uncertainties, and their unknown values in the master and the slave systems are not the same. Besides, it is supposed that the base excitation is perturbed by a stochastic signal, which models the conditions of real-world applications of the problem. The main objective is to synchronize two chaotic gyros under the mentioned situations. It must be noted that due to white noise application the dynamical systems under study are indeed stochastic chaotic systems. To model noise signal, a standard Wiener process is utilized.

In the next three sections, a general method for synchronizing such systems is introduced.

3. Stochastic chaos synchronization with uncertain parameters

A stochastic chaotic system with the following equation is considered as the drive or master system:

$$x^{(n)} = f(\mathbf{x}, t) + h(\mathbf{x}, t)\dot{v}, \tag{3}$$

where $\mathbf{x} = (x, \dot{x}, \ldots, x^{(n-1)}) = (x_1, x_2, \ldots, x_n) \in \Re^n$ is the state vector, the superscripts $(n-1)$ and (n) depict the derivative of order $n-1$ and n, respectively, $f: \Re^n \times \Re^+ \to \Re$ and $h: \Re^n \times \Re^+ \to \Re$ are two nonlinear and sufficiently smooth functions, $x^{(n)} = \mathrm{d}^n x/\mathrm{d}t^n$, v is a standard Wiener process and $\dot{v} = \mathrm{d}v/\mathrm{d}t$. It is assumed that the system $x^{(n)} = f(\mathbf{x}, t)$ is itself a chaotic system.

The response or slave system that must be controlled for synchronization is given by

$$y^{(n)} = g(\mathbf{y}, t) + k(\mathbf{y}, t)\dot{w} + b(\mathbf{y}, t)u, \tag{4}$$

where $\mathbf{y} = (y, \dot{y}, \ldots, y^{(n-1)}) = (y_1, y_2, \ldots, y_n) \in \Re^n$ is the state vector, the superscripts $(n-1)$, (n), etc. depict the derivative of order $n-1$, n, etc. respectively. $u \in \Re$ is the control variable of the system, w is a standard white noise independent of v, g,k,b: $\Re^n \times \Re^+ \to \Re$ are sufficiently smooth functions, and $y^{(n)} = g(\mathbf{y}, t)$ shows chaotic behavior. It is assumed that f, g and b are unknown functions whose nominal known values are denoted by $\hat{f}, \hat{g}$ and $\hat{b}$. f, g, b, $\hat{f}, \hat{g}$ and $\hat{b}$ satisfy the following conditions:

$$\left|f(\mathbf{x}, t) - \hat{f}(\mathbf{x}, t)\right| < F(\mathbf{x}, t), \quad \left|g(\mathbf{y}, t) - \hat{g}(\mathbf{y}, t)\right| < G(\mathbf{y}, t), \tag{5}$$

$$0 < b_m(\mathbf{y}, t) < b(\mathbf{y}, t), \hat{b}(\mathbf{y}, t) < b_M(\mathbf{y}, t) \tag{6}$$

where F, G, b_m and b_M are known functions. The above conditions play an important role in deriving the sliding mode control [23]. Although due to parameter uncertainties the exact values of the functions are not known, some upper bounds of uncertainties are necessary. It is also assumed that all the state variables of the master and slave systems are available for control design. Besides, the functions $h(\mathbf{x},t)$ and $k(\mathbf{y},t)$ are bounded with known bounds M_h and M_k, i.e. $|h(\mathbf{x},t)| \leqslant M_h$ and $|k(\mathbf{y},t)| \leqslant M_k$. The synchronization problem is to design a controller u, which synchronizes the states of both the master and slave systems in such a way that the slave trajectories follow the master trajectories. In deterministic systems, the usual synchronization objective is declared as

$$\left\|\mathbf{x} - \mathbf{y}\right\|_2 \to 0 \quad \text{as} \quad t \to \infty, \tag{7}$$

where $\|\cdot\|_2$ is the Euclidian norm. But it must be noticed that if the above condition is completely satisfied, i.e. $\mathbf{x}(t) - \mathbf{y}(t) \equiv 0$, comparing Eqs. (3) and (4) gives

$$x^{(n)} - y^{(n)} = f(\mathbf{x}, t) - g(\mathbf{y}, t) + h(\mathbf{x}, t)\dot{v} - k(\mathbf{y}, t)\dot{w} - b(\mathbf{y}, t)u = 0. \tag{8}$$

So,

$$\begin{aligned} &f(\mathbf{x}, t) - g(\mathbf{y}, t) + h(\mathbf{x}, t)\dot{v} - k(\mathbf{y}, t)\dot{w} - b(\mathbf{y}, t)u = 0 \\ &\quad \Rightarrow u = b^{-1}(\mathbf{y}, t)[f(\mathbf{x}, t) - g(\mathbf{y}, t) + h(\mathbf{x}, t)\dot{v} - k(\mathbf{y}, t)\dot{w}]. \end{aligned} \tag{9}$$

This implies that u directly depends on the white Gaussian noises and so it is not a Markov control, i.e. an accessible signal, so the objective of Eq. (7) cannot be achieved in our problem. Therefore, the synchronization goal must be modified as it is described in the next section.

4. Synchronizing control design

Defining the error vector $\mathbf{e} = \mathbf{y} - \mathbf{x}$ and subtracting Eq. (3) from Eq. (4) one gets the error dynamics as

$$e^{(n)} = g(\mathbf{y}, t) - f(\mathbf{x}, t) + b(\mathbf{y}, t)u + k(\mathbf{y}, t)\dot{w} - h(\mathbf{x}, t)\dot{v}. \tag{10}$$

Eq. (10) is a stochastic differential equation, which is expressed in the following differential form:

$$\mathrm{d}e^{(n-1)} = [g(\mathbf{y}, t) - f(\mathbf{x}, t) + b(\mathbf{y}, t)u]\,\mathrm{d}t + k(\mathbf{y}, t)\,\mathrm{d}w - h(\mathbf{x}, t)\,\mathrm{d}v. \tag{11}$$

The above equation is indeed an Ito stochastic differential form. Due to some technicalities in the definition of Wiener process, the differential form of Eq. (11) is usually used instead of Eq. (10). Here some important remarks that are frequently used in studying the stochastic systems are reviewed.

Remark 1. For manipulating stochastic Equation (11) and controller design, the convergence in the L^2 norm, i.e. the mean square norm, must be considered instead of the Euclidian norm. The mean square norm is defined as

$$\|\mathbf{e}\| = \left(E[\mathbf{e}^{\mathrm{T}}\mathbf{e}]\right)^{1/2}, \tag{12}$$

where $E[.]$ is the expected value function.

Remark 2. Regarding the definition of the Wiener process and independency of w and v [21], we have

$$\begin{gathered} E[k\,\mathrm{d}w] = E[h\,\mathrm{d}v] = E[k\,\mathrm{d}w h\,\mathrm{d}v] = 0, \\ E[h\,\mathrm{d}v]^2 = [h(\mathbf{x}, t)]^2\,\mathrm{d}t, \quad E[k\,\mathrm{d}w]^2 = [k(\mathbf{y}, t)]^2\,\mathrm{d}t. \end{gathered} \tag{13}$$

Again $E[.]$ is the expected value or mean value function. The above equations are indeed the main properties on which the Wiener process is based [21]. The equations in the first line of Eq. (13) are obtained according to the facts that the Wiener process generates a zero-mean noise, and $\dot{w}$ and $\dot{v}$ are two independent processes, i.e. their correlation is zero. The equations in the second line are based on the variance property of the Wiener process, which is linearly dependent on time. Eqs. (13) in the integral form are written as

$$\begin{aligned} &E\left[\int_{t_1}^{t_2} k\,\mathrm{d}w(s)\right] = E\left[\int_{t_1}^{t_2} h\,\mathrm{d}v(s)\right] = 0, \\ &E\left[\int_{t_1}^{t_2} h(\mathbf{x}(s), s)\,\mathrm{d}v\right]^2 = E\left[\int_{t_1}^{t_2} h^2(\mathbf{x}(s), s)\,\mathrm{d}s\right], \quad E\left[\int_{t_1}^{t_2} k(\mathbf{y}(s), s)\,\mathrm{d}w\right]^2 = E\left[\int_{t_1}^{t_2} k^2(\mathbf{y}(s), s)\,\mathrm{d}s\right]. \end{aligned} \tag{14}$$

The last two relations are called the Ito isometry.

Remark 3. Assuming that $z = z(\mathbf{x},t)$ is a twice-differentiable scalar function, we have [21]

$$\mathrm{d}z = \frac{\partial z}{\partial t}\,\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 z}{\partial x_i \partial x_j}\,\mathrm{d}x_i\,\mathrm{d}x_j, \tag{15}$$

where x_i is the ith element of vector $\mathbf{x}$, and $\mathbf{x}$ is obtained from Eq. (3). In expanding the above equation note that [21]

$$\mathrm{d}t\,\mathrm{d}t = 0, \quad \mathrm{d}t\,\mathrm{d}v = 0, \quad \mathrm{d}v\,\mathrm{d}v = \mathrm{d}t. \tag{16}$$

Besides, since v and w are independent we have $\mathrm{d}v\,\mathrm{d}w = 0$.

Now we are ready to start the control design procedure.

5. Sliding mode control modification

Due to uncertainties on the system parameters, we use the sliding mode concept to design the synchronizing control. To this end, a sliding surface must be defined in the form of

$$S(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda\right)^{n-1} e(t) = \sum_{m=0}^{n-1} \binom{n-1}{m} e^{(n-1-m)} \lambda^m, \quad \binom{n-1}{m} = \frac{(n-1)!}{m!(n-1-m)!}, \tag{17}$$

where $\lambda > 0$ is a positive constant, therefore $S(t) = 0$ gives an exponentially stable dynamics for $e(t)$. $S = 0$ makes a surface in the phase space. It is called the sliding surface because if the error trajectory $e(t)$ lies on the sliding surface, it slides and converges to zero [23]. The goal of sliding mode control is to design u such that the error states converge to the exponentially stable structure of $S(t) = 0$. To this end a Lyapunov function is defined as

$$V(t) = \tfrac{1}{2} E[S(t)]^2. \tag{18}$$

The differential form of V is obtained as

$$\mathrm{d}V(t) = \tfrac{1}{2} E[\mathrm{d}(S^2(t))]. \tag{19}$$

Note that $S(t)$ is a stochastic process whose differential form is

$$\begin{aligned} \mathrm{d}S(t) &= \sum_{m=0}^{n-1} \binom{n-1}{m} e^{(n-m)}(t)\,\mathrm{d}t\lambda^m \\ &= \sum_{m=1}^{n-1} \binom{n-1}{m} e^{(n-m)}(t)\,\mathrm{d}t\lambda^m + \mathrm{d}e^{(n-1)}. \end{aligned} \tag{20a}$$

Now substituting Eq. (11) into Eq. (20a) the following differential form is obtained:

$$\mathrm{d}S(t) = \sum_{m=1}^{n-1} \binom{n-1}{m} e^{(n-1-m)}(t)\lambda^m + [g(\mathbf{y},t) - f(\mathbf{x},t) + b(\mathbf{y},t)u]\,\mathrm{d}t + k(\mathbf{y},t)\,\mathrm{d}w - h(\mathbf{x},t)\,\mathrm{d}v. \tag{20b}$$

Using the property of Eq. (15) we have

$$\mathrm{d}\big(S^2(t)\big) = 2S(t)\,\mathrm{d}S(t) + \mathrm{d}S(t)\,\mathrm{d}S(t). \tag{21}$$

Substituting Eq. (20) into Eq. (21) and using Eq. (13), the differential form $\mathrm{d}V$ is obtained as

$$\begin{aligned} \mathrm{d}V(t) = E\left[S(t)\left(\sum_{m=1}^{n-1} \binom{n-1}{m} e^{(n-1-m)}(t)\lambda^m + g(\mathbf{y},t) - f(\mathbf{x},t) + b(\mathbf{y},t)u\right)\mathrm{d}t\right] + \frac{1}{2}E\left[k^2(\mathbf{y},t)\right]\mathrm{d}t \\ + \frac{1}{2}E\left[h^2(\mathbf{x},t)\right]\mathrm{d}t. \end{aligned} \tag{22}$$

So,

$$\dot{V} = E\left[S(t)\left(\sum_{m=1}^{n-1} \binom{n-1}{m} e^{(n-1-m)}(t)\lambda^m + g(\mathbf{y},t) - f(\mathbf{x},t) + b(\mathbf{y},t)u\right)\right] + \frac{1}{2}E\left[k^2(\mathbf{y},t)\right] + \frac{1}{2}E\left[h^2(\mathbf{x},t)\right]. \tag{23}$$

Now let the control action be

$$u = -\frac{1}{b_m(\mathbf{y},t)}\left[\hat{g}(\mathbf{y},t) - \hat{f}(\mathbf{x},t) + \sum_{m=1}^{n-1} \binom{n-1}{m} e^{(n-m)}\lambda^m + K\,\mathrm{sign}(S(t)) + \theta S(t)\right], \tag{24}$$

where $\theta > 0$ is a positive constant and K must satisfy the following condition:

$$K \geqslant \left\{ G(\mathbf{y},t) + F(\mathbf{x},t) + \left(\frac{b_M}{b_m} - 1\right)\left(\left|\hat{g}(\mathbf{y},t) - \hat{f}(\mathbf{x},t)\right| + \left|\sum_{m=1}^{n-1} \binom{n-1}{m} e^{(n-m)}\lambda^m\right|\right)\right\}. \tag{25}$$

Substituting Eq. (24) into Eq. (23), considering the condition of Eq. (25) and manipulating some calculations yield

$$\dot{V}(t) \leqslant -\theta E\left[S^2(t)\right] + \tfrac{1}{2}E\left[k^2(\mathbf{y},t) + h^2(\mathbf{x},t)\right]. \tag{26}$$

Regarding the upper bounds of $k(\mathbf{y}, t)$ and $h(\mathbf{x}, t)$, i.e. M_k and M_h, from Eq. (26) it is concluded that the following region of the error phase space is an attracting set for $\mathbf{e}(t)$:

$$\Omega = \left\{ \mathbf{e} \middle| E[S^2] < \frac{M_k^2 + M_h^2}{\theta} \right\}, \tag{27}$$

where $\mathbf{e} = (e, \dot{e}, \ldots, e^{(n-1)})$. It means that

$$E\left[S^2(t)\right] < \frac{M_k^2 + M_h^2}{\theta} \quad \text{as} \quad t \to \infty. \tag{28}$$

To show the attraction property of Ω, i.e. Eq. (28), two steps must be accomplished. At first it must be proved that when the error trajectories are outside the Ω set, they converge to it. Then it must be proved that when the error trajectories are inside the Ω set, they cannot escape from it. To this end, assume that at time t, $S(t)$ is outside the Ω set, then it is concluded that

$$E\left[S^2(t)\right] \geqslant \frac{M_k^2 + M_h^2}{\theta}. \tag{29}$$

Now considering Eq. (26) it is obtained that, in this case $\dot{V}(t) < 0$, so when $\mathbf{e}(t)$ is outside the Ω set, V is a descending function. Since $V(t) = \frac{1}{2}E[S^2(t)]$ and $\dot{V}(t) < 0$, $E[S^2(t)]$ decreases until $\mathbf{e}(t)$ enters the Ω set finally.

On the other hand if at time t, $S(t)$ is inside the Ω set, it cannot exit the Ω set, because if $S(t)$ wants to exit Ω then it must pass the following region in the phase space:

$$\frac{M_k^2 + M_h^2}{2\theta} < E\left[S^2(t)\right] < \frac{M_k^2 + M_h^2}{\theta} \tag{30}$$

and considering Eq. (26) and regarding this matter that M_k and M_h are some upper bounds for $k(\boldsymbol{y}, t)$ and $h(\boldsymbol{x}, t)$, it is obtained that in the region indicated by Eq. (30), $\dot{V}(t) < 0$ and hence $V(t)$ descends and $\mathbf{e}(t)$ cannot escape from the Ω set. Therefore, Ω is an attracting set for $S(t)$, and the correctness of Eq. (28) is proved. This property of the control action (24), i.e. the attraction of Ω, is used in the next section to show the synchronization property of the controller.

6. Synchronization property of the controller

Using the definition of $S(t)$ and Eq. (28), it can be written as

$$S(t) = \sum_{m=0}^{n-1} \binom{n-1}{m} \lambda^m e^{(n-1-m)}(t) = \delta(t). \tag{31}$$

From the previous section it is clear that for sufficiently large t, we have

$$E\left[\delta^2(t)\right] < \frac{M_k^2 + M_h^2}{\theta}. \tag{32}$$

Note that $\delta(t)$ is a continuous disturbance and is not a white noise. Now let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\lambda^{n-1}\binom{n-1}{n-1} & -\lambda^{n-2}\binom{n-1}{n-2} & \cdots & \cdots & -\lambda\binom{n-1}{1} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{\eta}(t) = \begin{bmatrix} e(t) \\ \dot{e}(t) \\ \vdots \\ e^{(n-2)}(t) \\ e^{(n-1)}(t) \end{bmatrix}. \tag{33}$$

Using the above definitions, Eq. (31) can be rewritten as

$$\dot{\boldsymbol{\eta}}(t) = \mathbf{A}\boldsymbol{\eta}(t) + \mathbf{B}\delta(t). \tag{34}$$

Since $\mathbf{A}$ is a Hurwitz matrix, i.e. all of its eigenvalues have negative real parts, the following Lyapunov equation has a symmetric positive definite answer, $\mathbf{P}$:

$$\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} = -\mathbf{I}, \tag{35}$$

where $\mathbf{I}$ is the identity matrix. Now define a Lyapunov function as

$$V(t) = \tfrac{1}{2}E\left[\boldsymbol{\eta}^{\mathrm{T}}(t)\mathbf{P}\boldsymbol{\eta}(t)\right] \tag{36}$$

and regarding the continuity of $\boldsymbol{\eta}(t)$, consider the derivative of V along $\boldsymbol{\eta}(t)$:

$$\begin{aligned}\dot{V}(t) &= E\left[\dot{\boldsymbol{\eta}}^{\mathrm{T}}(t)\mathbf{P}\boldsymbol{\eta}(t) + \boldsymbol{\eta}^{\mathrm{T}}(t)\mathbf{P}\dot{\boldsymbol{\eta}}(t)\right] \\ &= E\left[\left(\boldsymbol{\eta}^{\mathrm{T}}(t)\mathbf{A}^{\mathrm{T}} + \delta(t)\mathbf{B}^{\mathrm{T}}\right)\mathbf{P}\boldsymbol{\eta}(t) + \boldsymbol{\eta}^{\mathrm{T}}(t)\mathbf{P}(\mathbf{A}\boldsymbol{\eta}(t) + \mathbf{B}\delta(t))\right] \\ &= -E\left[\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{\eta}(t)\right] + 2E\left[\left(\mathbf{B}^{\mathrm{T}}\mathbf{P}\boldsymbol{\eta}(t)\right)\delta(t)\right].\end{aligned} \tag{37}$$

Define

$$\Delta = \left\{\boldsymbol{\eta}\,\middle|\, \left\|\boldsymbol{\eta}(t)\right\| < 2\left\|\mathbf{B}^{\mathrm{T}}\mathbf{P}\right\|\left(\frac{M_k^2 + M_h^2}{\theta}\right)^{1/2}\right\}. \tag{38}$$

The above set is an attracting set for $\boldsymbol{\eta}(t)$, since using the Cauchy–Schwarz inequality we have

$$\begin{aligned}\left|E\left[\mathbf{B}^{\mathrm{T}}\mathbf{P}\boldsymbol{\eta}(t)\delta(t)\right]\right| &\leqslant \left\|\mathbf{B}^{\mathrm{T}}\mathbf{P}\boldsymbol{\eta}(t)\right\|\left\|\delta(t)\right\| \\ &\leqslant \left\|\mathbf{B}^{\mathrm{T}}\mathbf{P}\right\|\left\|\boldsymbol{\eta}(t)\right\|\left(\frac{M_h^2 + M_k^2}{\theta}\right)^{1/2}.\end{aligned} \tag{39}$$

If $\boldsymbol{\eta}(t)$ is out of the Δ set, we have

$$\begin{aligned}\left\|\boldsymbol{\eta}(t)\right\| > 2\left\|\mathbf{B}^{\mathrm{T}}\mathbf{P}\right\|\left(\frac{M_h^2 + M_k^2}{\theta}\right)^{1/2} &\Rightarrow -\left\|\boldsymbol{\eta}(t)\right\| + \left\|\mathbf{B}^{\mathrm{T}}\mathbf{P}\right\|\left(\frac{M_h^2 + M_k^2}{\theta}\right)^{1/2} < 0 \\ &\Rightarrow -E\left[\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{\eta}(t)\right] + E\left[\mathbf{B}^{\mathrm{T}}\mathbf{P}\boldsymbol{\eta}(t)\delta(t)\right] = \dot{V}(t) < 0.\end{aligned} \tag{40}$$

Thus, $\boldsymbol{\eta}(t)$ must be attracted by Δ. Inside the Δ set, $\boldsymbol{\eta}(t)$ cannot exit Δ, because in this case $\boldsymbol{\eta}(t)$ must pass the region defined by

$$\left\|\mathbf{B}^{\mathrm{T}}\mathbf{P}\right\|\left(\frac{M_h^2 + M_k^2}{\theta}\right)^{1/2} < \left\|\boldsymbol{\eta}(t)\right\| < 2\left\|\mathbf{B}^{\mathrm{T}}\mathbf{P}\right\|\left(\frac{M_h^2 + M_k^2}{\theta}\right)^{1/2} \tag{41}$$

and in this region, $\dot{V}(t) < 0$.

Therefore Δ is an attracting set, and for large enough t, $\boldsymbol{\eta}(t)$ satisfies the relation below:

$$E\left[\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{\eta}(t)\right] < 4\left\|\mathbf{B}^{\mathrm{T}}\mathbf{P}\right\|^2\frac{M_h^2 + M_k^2}{\theta}. \tag{42}$$

In addition from Eq. (31) it is obtained that

$$e^{(n-1)}(t) = -\sum_{m=1}^{n-1}\binom{n-1}{m}\lambda^m e^{(n-1-m)}(t) + \delta(t). \tag{43}$$

The above equation yields

$$\left\|e^{(n-1)}(t)\right\| \leqslant \sum_{m=1}^{n-1}\binom{n-1}{m}\lambda^m\left\|e^{(n-1-m)}(t)\right\| + \left\|\delta(t)\right\|. \tag{44}$$

So,

$$\left\|e^{(n-1)}(t)\right\| \leqslant \left(\sum_{m=1}^{n-1}\binom{n-1}{m}\lambda^m\left\|\mathbf{B}^{\mathrm{T}}\mathbf{P}\right\|+1\right)\left(\frac{M_h^2+M_k^2}{\theta}\right)^{1/2}. \tag{45}$$

Eqs. (42) and (44) show that the mean square norm of the tracking error can be smaller than any positive number if θ is chosen to be sufficiently large.

7. Stochastic chaos synchronization in two chaotic gyros

In this section, the proposed synchronization algorithm is applied to the problem of gyro synchronization mentioned in Section 2. The dynamic equations of the master and slave systems are given, respectively, by

$$\ddot{x} = -\alpha^2\frac{(1-\cos x)^2}{\sin^3 x} - c_1\dot{x} - c_2\dot{x}^3 + \beta\sin x + (f\sin\omega t + \mu\dot{v})\sin x, \tag{46}$$

$$\ddot{y} = -\alpha_s^2\frac{(1-\cos y)^2}{\sin^3 y} - c_{1s}\dot{y} - c_{2s}\dot{y}^3 + \beta_s\sin y + (f_s\sin\omega t + \mu_s\dot{w})\sin y + (1+|y|)u, \tag{47}$$

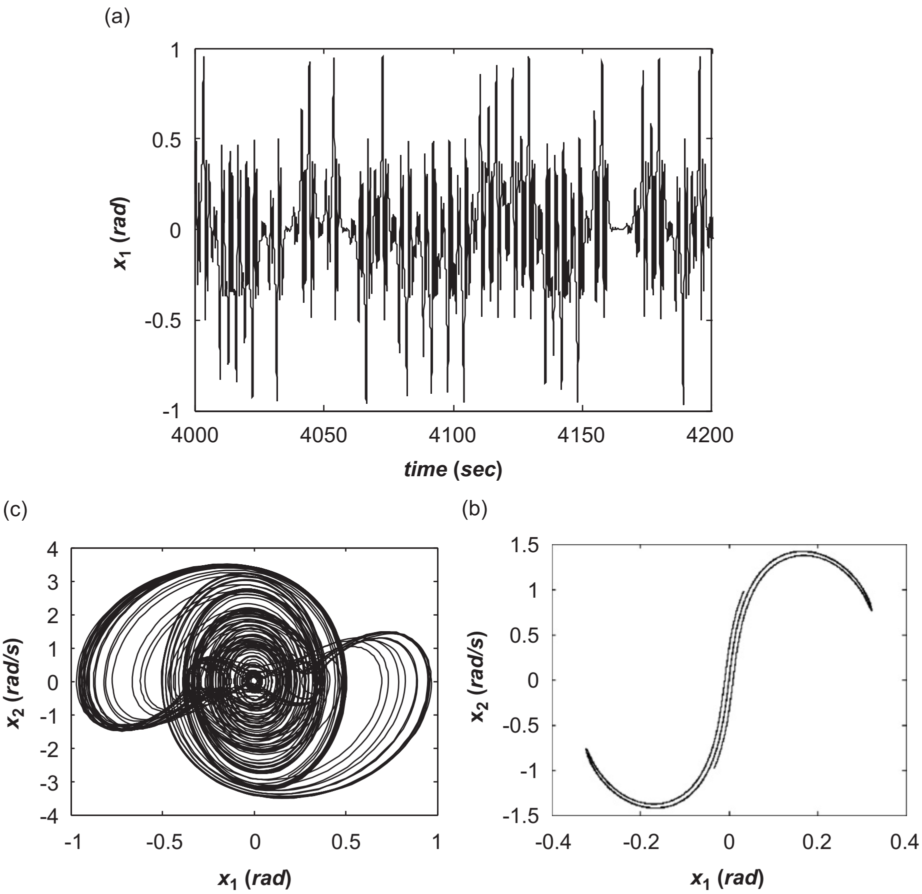

Fig. 1. (a) Time series of $x_1(t)$ strange attractors: (b) on the Poincare map and (c) in the phase space.

where $\dot{v}$ and $\dot{w}$ are two independent white Gaussian noises, μ and μ_s are two real constants and u is the control action. According to Eqs. (3) and (4) the functions $f(.)$, $g(.)$, $h(.)$, $k(.)$ and $b(.)$ are

$$\begin{aligned} f(\mathbf{x},t) &= -\alpha^2\frac{(1-\cos x)^2}{\sin^3 x} - c_1\dot{x} - c_2\dot{x}^3 + \beta\sin x + f\sin\omega t\sin x,\\ h(\mathbf{x},t) &= \mu\sin x,\\ g(\mathbf{y},t) &= -\alpha_s^2\frac{(1-\cos y)^2}{\sin^3 y} - c_{1s}\dot{y} - c_{2s}\dot{y}^3 + \beta_s\sin y + f_s\sin\omega t\sin y,\\ k(\mathbf{y},t) &= \mu_s\sin y,\\ b(\mathbf{y},t) &= 1+|y|, \end{aligned} \tag{48}$$

where $\mathbf{x} = (x_1 \quad x_2) = (x \quad \dot{x})$ and $\mathbf{y} = (y_1 \quad y_2) = (y \quad \dot{y})$. For $\alpha^2 = 100$, $\beta = 1$, $c_1 = 0.5$, $c_2 = 0.05$, $\omega = 2$, $f = 35.5$, $\alpha_s^2 = 94$, $\beta_s = 1.2$, $c_{1s} = 0.45$, $c_{2s} = 0.04$, $f_s = 34$, $u = 0$ and $\mu = \mu_s = 0$, the master and the slave systems show chaotic responses. Fig. 1 shows the chaotic behavior of the master system in the phase space and the Poincare map of $t = 2n\pi/\omega$, $n = 1,2,\ldots$.

Due to system uncertainties the nominal values of parameters that are assumed to be known are denoted by $\hat{\alpha}^2 = 110$, $\hat{\beta} = 1.4$, $\hat{c}_1 = 0.7$, $\hat{c}_2 = 0.06$, $\hat{f} = 37$, $\hat{\alpha}_s^2 = 90$, $\hat{\beta}_s = 0.8$, $\hat{c}_{1s} = 0.34$, $c_{2s} = 0.046$ and $\hat{f}_s = 33$.

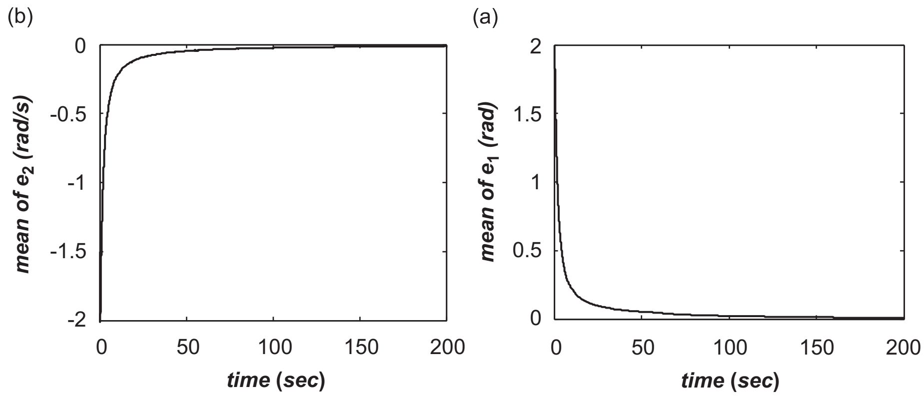

Fig. 2. Mean values of synchronization errors: (a) $E(e_1)$ and (b) $E(e_2)$.

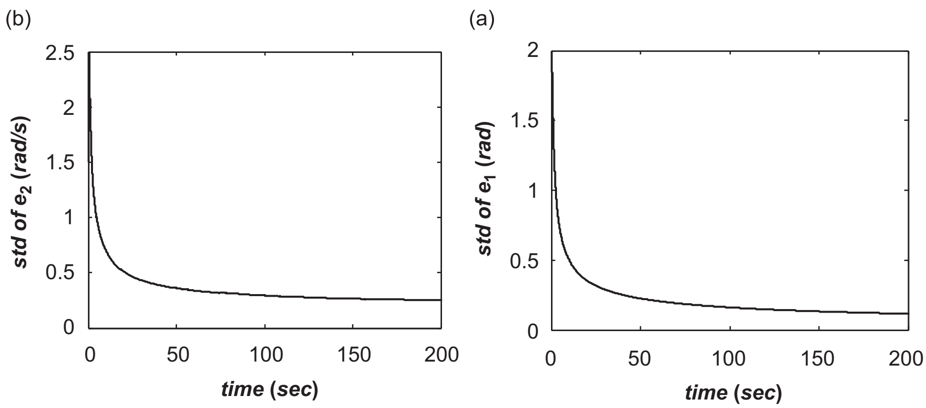

Fig. 3. Mean square norms (standard deviations) of synchronization errors: (a) $[E(e_1^2)]^{1/2}$ and (b) $[E(e_2^2)]^{1/2}$.

Thus the nominal functions $\hat{f}(.), \hat{g}(.), \hat{b}(.)$ are obtained as

$$\begin{aligned}
\hat{f}(\mathbf{x}, t) &= -\hat{\alpha}^2 \frac{(1-\cos x)^2}{\sin^3 x} - \hat{c}_1\dot{x} - \hat{c}_2\dot{x}^3 + \hat{\beta}\sin x + \hat{f}\sin\omega t \sin x, \\
\hat{g}(\mathbf{y}, t) &= -\hat{\alpha}_s^2 \frac{(1-\cos y)^2}{\sin^3 y} - \hat{c}_{1s}\dot{y} - \hat{c}_{2s}\dot{y}^3 + \hat{\beta}_s\sin y + \hat{f}_s\sin\omega t \sin y, \\
\hat{b}(\mathbf{y}, t) &= 1 + 0.5|y|.
\end{aligned} \tag{49}$$

So the bound functions $F(.)$, $G(.)$, $b_m(.)$ and $b_M(.)$ may be chosen as

$$\begin{aligned}
F(\mathbf{x}, t) &= 20\frac{(1-\cos x)^2}{|\sin^3 x|} + 0.5|\dot{x}| + 0.1|\dot{x}^3| + 7|\sin x|, \\
G(\mathbf{y}, t) &= 20\frac{(1-\cos y)^2}{|\sin^3 y|} + 0.5|\dot{y}| + 0.1|\dot{y}^3| + 7|\sin y|, \\
b_m(\mathbf{y}, t) &= 0.5 + 0.5|y|, \quad b_M(\mathbf{y}, t) = 2 + |y|.
\end{aligned} \tag{50}$$

Also the parameters μ and μ_s are set to 1 and in Eqs. (17) and (24), we set $\theta = 4$ and $\lambda = 1$. Simulation results of applying the proposed control law are shown in Figs. 2 and 3. It is seen that the synchronization objective is achieved such that the mean values of tracking errors converge to zero (Fig. 2) and the mean square norms of errors (standard deviations, std) converge to a narrow vicinity of zero (Fig. 3).

Synchronization error time series and time series of the master and slave states are shown in Figs. 4 and 5. As it is observed, although the states of the slave system converge to the states of the master system, they deviate randomly around the target in a way that the statements of Eqs. (42) and (45) are satisfied.

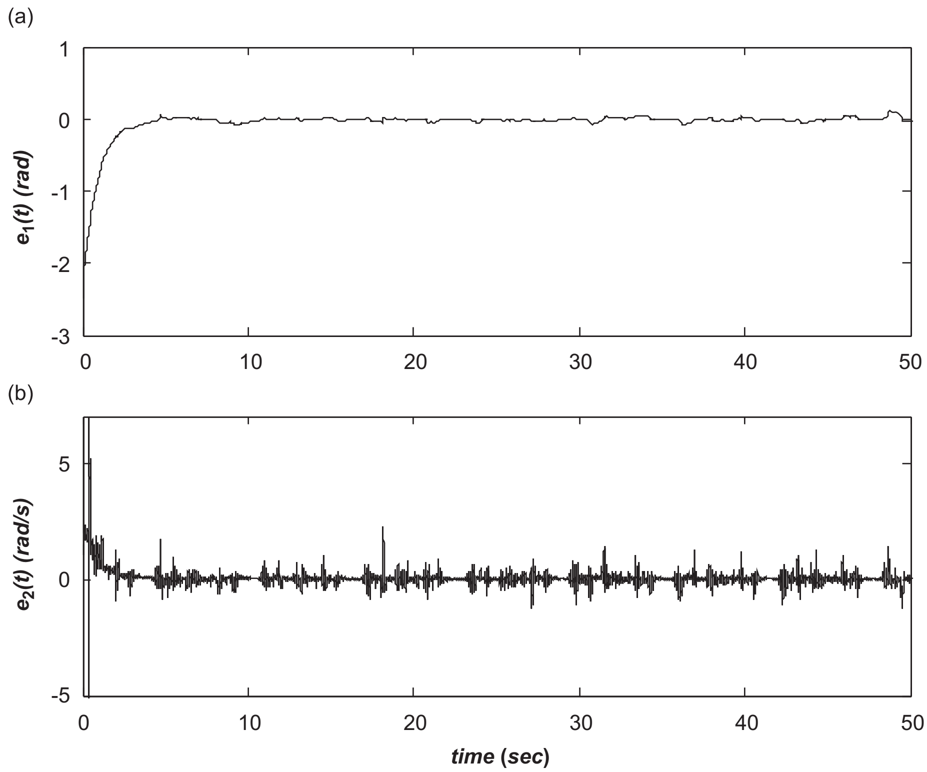

Fig. 4. Time series of synchronization errors: (a) $e_1(t)$ and (b) $e_2(t)$.

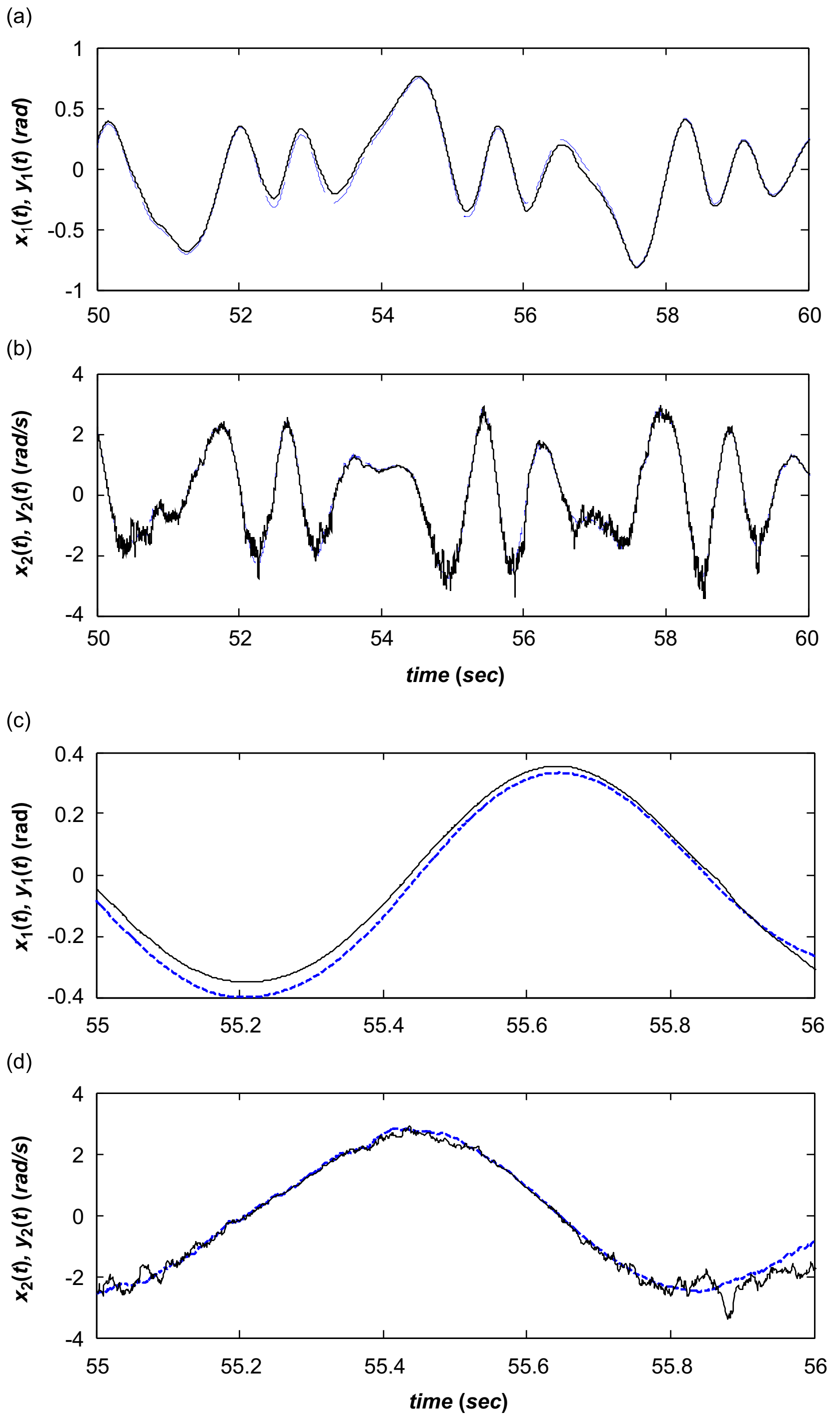

Fig. 5. Time series of the master states (continuous lines) and the slave states (discrete lines): (a) $x_1(t)$ and $y_1(t)$, (b) $x_2(t)$ and $y_2(t)$, (c) zoomed figure of (a), and (d) zoomed figure of (b).

8. Conclusions

In this paper synchronization of two stochastic chaotic gyros, which have different parameters, has been investigated. The stochastic chaos has been modeled through to the Ito differential form by adding white

Gaussian noises to the excitation signals. Using the concept of sliding mode control, a synchronizing controller has been designed to make the error trajectories converge in the mean square to the sliding surface. It has been shown that all of the state variables of the response system converge to the states of the drive system such that the tracking error variance finally falls into an arbitrarily small neighborhood of zero.

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